Module 1 :

The equation of “continuity”

Lecture 1:

Equation of Continuity

Advanced Heat and Mass Transfer: Modules

1. **THE EQUATION OF "CONTINUITY": Lectures 1-6**

    (i) Overall Mass Balance

    (ii) Momentum Balance

    (iii) Energy Balance

    (iv) Special Mass Balance

    (v) Equation for the fluxes

2. **"DIFFUSIVE" HEAT AND MASS TRANSFER: Lectures 7-20**

    (i) Steady and Unsteady/One and Multiple Dimensions

    (ii) Mass Transfer with Chemical Reaction

    (iii) Perturbation Techniques

    (iv) Moving Boundary Problems

    (v) Simultaneous Heat and Mass Transfer

3. **"CONVECTIVE" HEAT AND MASS TRANSFER: Lectures 21-32**

    (i) Flow Inside Ducts

    (ii) Dispersion

    (iii) Laminar Boundary Layers

    (iv) Mass Transfer with Chemical Reactions

(v) Asymptotic Methods

(vi) Simultaneous Momentum, Heat and Mass Transfer

(vii) Natural Convection

4. **MULTICOMPONENT TRANSPORT: Lectures 33-37**

(i) Binary Systems

(ii) Muti-component Flux Equations

(iii) Thermal Diffusion

(iv) Dimensional Analysis

5. **MASS TRANSFER IN TURBULANT FLOWS: Lectures 38-41**

(i) Time Averaging and Eddy Viscosity

(ii) Universal Velocity

(iii) Mass Transfer in Turbulent Pipe Flow

Reference Books

1. Bird, R.B., Stewart, W.E. and Lightfoot, E.N., "Transport Phenomenon", Wiley (1960).
2. Carslaw, H.S. and Jaeger, J.C., "Conduction of heat in Solids", (2nd ed) Oxford (1975).
3. Slattery, J., Momentum, Energy and Mass Transfer in Continua", (2nd ed) Krueger (1981).

Goals of the Course

- **To relate mathematical symbols to physical reality**

- **To review several classic problems**

- **To show examples of how to approach the unknown**

THE EQUATION OF "CONTINUITY"

The Continuum Approximation

Field variables (e.g. velocity) at a "point" are spatial averages over a small volume V around that point, where V has to be such that

$$\ell << V^{1/3} << D, \qquad (1.1a)$$

where, ℓ is a characteristic **microscopic** length scale, which can be of molecular dimensions or the distance between molecules in a gas or the particle size in a solid/fluid two-phase system, and D is a characteristic **macroscopic** length scale. For example, for flow in a pipe, D can be the pipe diameter.

The continuum approximation considers the fluids to be continuous. Thus, the fluid properties such as temperature, pressure, density and velocity of the fluid are taken to be well defined at *infinitely* small points (i.e. at microscopic level), defining a reference element of volume, let's call this volume **REV**, at the geometric order of the distance between the two adjacent molecules of fluid. Properties are assumed to vary continuously from one point to another, and are averaged over the volume **REV**. The fact that the fluid is made up of discrete molecules is ignored.

Eulerian and Lagrangian coordinates

Eulerian Coordinate: in this system the independent variables are x, y, z and t or x_i (i=1, 2, 3) and t. This is a fixed coordinate system. The basic conservation equation are in the Eulerian frame, R = R (x_i, t).

In the Lagrangian frame, attention is fixed on a particular mass of fluid as it flows, R = R (x_i^o, t), where the coordinate x_i^o specifies which fluid element is being considered.

Material Derivative

Consider a variable α such that

$$\alpha = \alpha(x_i, t) \qquad (1.1b)$$

Then the total differential of α can be expressed as

$$\delta\alpha = \frac{\partial\alpha}{\partial t}\delta t + \frac{\partial\alpha}{\partial x_i}\delta x_i \qquad (1.1c)$$

Division by a time differential δt leads to the following expression:

$$\frac{\delta\alpha}{\delta t} = \frac{\partial\alpha}{\partial t} + \frac{\partial\alpha}{\partial x_i}\frac{\delta x_i}{\delta t} \qquad (1.1d)$$

After taking the limit δt→0, we obtain for the material derivative

$$\frac{D\alpha}{Dt} = \frac{\partial\alpha}{\partial t} + v_i\frac{\partial\alpha}{\partial x_i} \qquad (1.1e)$$

where, $v_i = \left[\dfrac{\partial x_i}{\partial t}\right]_{\text{in the limit of } \delta t \to 0}$ is the fluid velocity in direction i,

$\dfrac{D\alpha}{Dt}$ is called the Material Derivative or Lagrangian Derivative in time and $\dfrac{\partial\alpha}{\partial t}$ is the Eulerian Derivative in time.

The Material derivative or Lagrangian time derivative represents in total change in α as seen by an **observer** who is moving with a particular fluid element. In the Lagrangian frame, we observe the particle for a time δt as it flows. The position of a particle changes by δx_i while α changes as $\delta\alpha$.

Time derivatives

The time derivative is a derivative of a function with respect to time. It implies the rate of change of value of a function with respect to time **t**.

Partial time derivative $\frac{\partial c}{\partial t}$ (at a point)

Total time derivative

By dividing by dt, the total differential can be written as *total time derivative*

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x}\frac{dx}{dt} + \frac{\partial c}{\partial y}\frac{dy}{dt} + \frac{\partial c}{\partial z}\frac{dc}{dt} \qquad (1.1f)$$

This expression represents the change in the time of the function c i.e., $(\partial c/\partial t)$ as we move about with arbitrary velocities in the coordinate directions i.e., (dx/dt, dy/dt and dz/dt).

Substantial Derivative or Material Derivative

If we constrain the motion to follow the motion of the individual fluid particles, we obtain the Substantial Derivative or Material Derivative (also known as Convective Derivative) given by

$$\text{Substantial time Derivative or Convective Derivative}: \frac{Dc}{dt} = \frac{\partial c}{\partial t} + U_x\frac{\partial c}{\partial x} + U_y\frac{\partial c}{\partial y} + U_z\frac{\partial c}{\partial z} \qquad (1.1g)$$

where U_x, U_y and U_z are the components of the local fluid velocity $\underline{U}$ in x, y and z directions, respectively.

The Mass Continuity Equation

The continuity equation is an overall mass balance about a control volume. Consider a volume element of volume V fixed in space as shown in figure below. Here the volume V is bounded by a surface S with outward unit normal vector $\underline{n}$

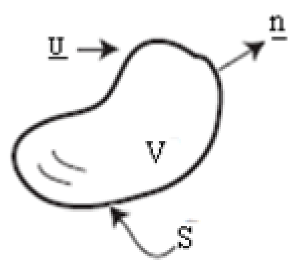

Fig. 1.1 Control Volume for Mass Continuity equation

$$(\text{Accumulation of mass inside V}) = (\text{The Net influx of mass through surface S})$$

$$\int_V \frac{\partial \rho}{\partial t} dV = -\oint_S \rho \underline{U} \bullet \underline{n} \, dS \qquad (1.2)$$

Here, ρ is the mass density. The minus sign (-) in front of the integral is because of the choice of $\underline{n}$ pointing outwards.

The **Divergence Theorem (Gauss)** for a vector field A gives

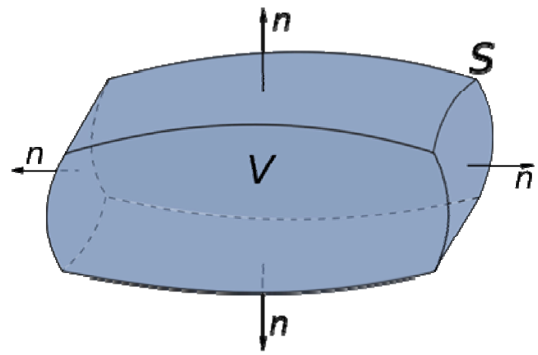

Fig.1.2. Gauss Divergence theorem

$$\int_V (\nabla \bullet \underline{A}) dV = \oint_S (\underline{A} \bullet \underline{n}) dS \qquad (1.3a)$$

Where ∇ is gradient (a vector) and can be expressed as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \qquad (1.3b)$$

In equation (1.3a), the left hand side is the volume integral over the volume V, the right hand side is the surface integral over the boundary of the volume V. The closed manifold dV is quite generally the boundary of V oriented by outward-pointing normals and $\underline{n}$ is the outward pointing unit normal field of the boundary dV.

For $\underline{A} = \rho \underline{U}$, equation (1.2) becomes

$$-\int_V \nabla \bullet (\rho \underline{U}) dV = -\oint_S (\rho \underline{U} \bullet \underline{n}) dS \qquad (1.3c)$$

Substitute equation (1.3c) on the right hand side of equation (1.2) we get,

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla \bullet (\rho \underline{U}) dV \qquad (1.3d)$$

or

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho \underline{U} \right) \right] dV = 0 \tag{1.3e}$$

Equation (1.3e) is known as Mass Continuity equation.

Since this equation must hold for arbitrary *V*, **Mass Continuity Equation** becomes

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho \underline{U} \right) = 0} \tag{1.4a}$$

or

$$\frac{D\rho}{Dt} + \rho \left(\nabla \bullet \underline{U} \right) = 0 \tag{1.4b}$$

From equation (1.1g), we know that

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \left(\underline{U} \bullet \nabla \rho \right) \tag{1.4c}$$

Incompressible fluids: Incompressible fluids are those fluids that do not exhibit any variation in **density** either in space or time. Therefore for incompressible fluids $\nabla \rho = 0$ and $\frac{\partial \rho}{\partial t} = 0$.

If ρ is constant (for incompressible fluids) in space and time, then the equation of continuity for incompressible fluids becomes

$$\nabla \bullet \underline{U} = 0 \tag{1.4d}$$

Equation (1.4b) can also be written as:

$$\frac{1}{V} \frac{DV}{Dt} = \left(\nabla \bullet \underline{U} \right) \tag{1.4e}$$

where, $V = \frac{1}{\rho}$ and $\frac{1}{V} \frac{DV}{Dt}$ is the rate of dilation of the fluid.

Application: We can apply the principle of continuity to pipes with cross sections which changes along their length. See Fig 1.3 below.

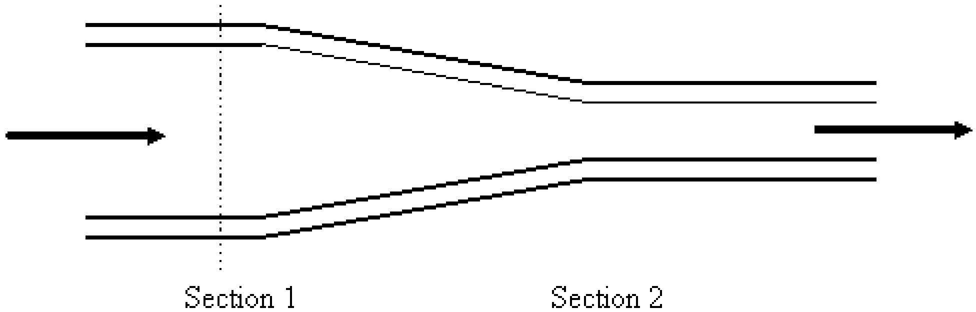

Fig.1.3. fluid flowing through convergent-divergent section of the pipe

A liquid is flowing from left to right and the pipe is narrowing in the same direction. By the continuity principle, the *mass flow rate* must be the same at each section.